*Statistics and Applications*
Volume 6, Nos.1 & 2, 2008 (New Series), pp.119-123

# A new class of orthogonal Latin hypercubes

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**Abstract**

In this paper, we develop a new class of orthogonal Latin hypercubes (OLHs) based on Latin squares. These OLHs have $n = 2^{r+1} + 1$ rows and $k = 2^r$ columns $(r = 1, 2, \ldots)$. For a given number of runs, our OLH vastly increases the numbers of orthogonal columns of OLHs in Ye (1998) and Cioppa & Lucas (2007).

*Key words:* Computer experiments; Latin squares.

## 1 Introduction

Latin hypercubes (LHs) were introduced by McKay, Beckman and Conover (1979) for computer experiments. An $n \times k$ LH can be represented by a design matrix $D_{n \times k}$ with $n$ rows (runs) and $k$ columns (factors), each of which includes $n$ uniformly spaced levels. An LH is called an orthogonal LH (OLH) if each pair of columns of this LH has zero correlation. Ye (1998) introduced a class of OLHs for $n = 2^{r+1} + 1$ rows and $k = 2r$ columns $(r = 1, 2, \ldots)$ using permutation matrices. Cioppa & Lucas (2007) extended Ye's method by introducing new orthogonal columns to Ye's OLHs. For a given $r$, the number of columns in Cioppa & Lucas's OLHs is $1 + r + \binom{r}{2}$. In this paper we show how to construct OLHs with $n = 2^{r+1} + 1$ rows and $k = 2^r$ columns $(r = 1, 2, \ldots)$. This vastly increases the number of columns of OLHs in Ye (1998) and Cioppa & Lucas (2007).

# 2 Constructing OLHs by permutation matrices

Both methods of Ye (1998) and Cioppa & Lucas (2007) require three $q \times k$ $(k < q)$ matrices $M$, $S$, and $T$ with $q = 2^r$. The first column of $M$ is $e = (1, 2, \ldots, q)'$. This column and permutation matrices are used to generate the remaining $k-1$ columns of $M$. $S$ is a $\pm 1$ matrix. $T$ is the element-wise product of $M$ and $S$. The corresponding $n \times k$ OLH is $[T' \; 0' \; -T']'$ where $0_{1\times k}$ is a row vector of 0's.

The $q \times q$ permutation matrix $A_i$ $(i = 1, 2, \ldots, r)$ is constructed as:

$$A_i = I \otimes \ldots \otimes I \otimes R \otimes \ldots \otimes R, \tag{1}$$

where $I$ is the $2 \times 2$ identity matrix , $R = [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}]$ and $\otimes$ is the Kronecker product. There are $r - i$ $I$'s and $i$ $R$'s in (1).

The matrix $M$ in Ye (1998) contains $k = 2r$ column vectors: $e$, $A_i e$ $(i = 1, 2, \ \ldots, r)$, and $A_i A_r e$ $(i = 1, 2, \ldots, r-1)$. The matrix $M$ in Cioppa & Lucas (2007), however, contains $k = 1 + r + \binom{r}{2}$ column vectors: $e$, $A_i e$ $(i = 1, 2, \ \ldots, r)$, and $A_i A_j e$ $(i = 1, 2, \ldots, r-1; \; j = i+1, \ldots, r)$. The matrix $S$ in the work of these authors corresponds to columns used to estimate the mean, main effects and 2-factor interactions of a $2^r$ factorial.

# 3 Constructing OLHs by latin squares

Our method requires three $q \times q$ matrices $M_r$, $S_r$ and $T_r$ with $q = 2^r$. $M_r$ is a Latin square of order $2^r$. Define $M_1$ as $[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}]$, $S_1$ as $[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}]$ and thus $T_1$ will become $[\begin{smallmatrix} 1 & 2 \\ 2 & -1 \end{smallmatrix}]$.

To construct $M_r$ we replace symbols $1, 2, \ldots$ of $M_{r-1}$ with matrices $[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}]$, $[\begin{smallmatrix} 3 & 4 \\ 4 & 3 \end{smallmatrix}]$, etc. $M_2$ will thus be:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

To construct the $\pm 1$ matrix $S_r$, we partition matrix $S_{r-1}$ as $[\begin{smallmatrix} P \\ Q \end{smallmatrix}]$ where $P$ and $Q$ are two matrices of the same size. $S_r$ is computed as

$[\begin{smallmatrix} S_{r-1} & R \\ S_{r-1} & -R \end{smallmatrix}]$ where $R = [\begin{smallmatrix} P \\ -Q \end{smallmatrix}]$. It can easily be shown that $S_r' S_r = S_r S_r' = 2^r I$. $S_2$ constructed this way is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

and $T_2$ becomes:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & -4 & 3 \\ 3 & 4 & -1 & -2 \\ 4 & -3 & 2 & -1 \end{pmatrix}$$

It can be verified that $T_3$ is:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & -1 & -4 & 3 & 6 & -5 & -8 & 7 \\ 3 & 4 & -1 & -2 & -7 & -8 & 5 & 6 \\ 4 & -3 & 2 & -1 & -8 & 7 & -6 & 5 \\ 5 & 6 & 7 & 8 & -1 & -2 & -3 & -4 \\ 6 & -5 & -8 & 7 & -2 & 1 & 4 & -3 \\ 7 & 8 & -5 & -6 & 3 & 4 & -1 & -2 \\ 8 & -7 & 6 & -5 & 4 & -3 & 2 & -1 \end{pmatrix}$$

The $(2^{r+1}+1) \times 2^r$ OLH can be constructed from $T_r$ in the same way that the OLH is constructed from $T$ (cf. Section 2). The Appendix displays the $33 \times 16$ OLH constructed by the method in this Section. Larger OLHs are available at `http://designcomputing.net/olh/`.

**Notes:**

1. It can be seen that the seven columns of the matrix $T$ used to construct the $17 \times 7$ OLH in Cioppa & Lucas (2007) are a subset of columns of our $T_3$ (some columns are with reverse signs).

2. Partition $M_r$ as $[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}]$ where $M_{11},\ M_{12}, M_{21}$ and $M_{22}$ are four $2^{r-1} \times 2^{r-1}$ matrices. It can be seen that $M_{11} = M_{22} = M_{r-1}$ and $M_{12} = M_{21} = M_{11} + 2^{r-1} J$ where $J$ is the $2^{r-1} \times 2^{r-1}$ matrix of 1's.

3. Let $T_{11}$ be the matrix formed by the first $2^{r-1}$ rows and $2^{r-1}$ columns of $T_r$. It can be seen that $T_{11} = T_{r-1}$.

## 4 Concluding remarks

In this paper, we show a time and space saving method of constructing OLHs. For given numbers of runs $n$ =17, 33, 65, 129, 257, 513 and 1025 (which corresponds to $r$=3, ..., 9), the maximum numbers of columns of OLHs in Ye (1998) are 6, 8, 10, 12, 14, 16 and 18 respectively. These numbers in Cioppa & Lucas (2007) are 7, 11, 16, 22, 29, 37 and 46 respectively (cf. Table 1 of Cioppa & Lucas (2007)). These numbers in our work are 8, 16, 32, 64, 128, 256 and 512. Thus our method greatly increases the numbers of columns in the constructed OLHs.

**Acknowledgments :** The author would like to thank Min-Qian Liu for valuable comments.

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# APPENDIX
## 33 × 16 Latin square based-OLH

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 2 | −1 | −4 | 3 | 6 | −5 | −8 | 7 | 10 | −9 | −12 | 11 | 14 | −13 | −16 | 15 |
| 3 | 4 | −1 | −2 | −7 | −8 | 5 | 6 | 11 | 12 | −9 | −10 | −15 | −16 | 13 | 14 |
| 4 | −3 | 2 | −1 | −8 | 7 | −6 | 5 | 12 | −11 | 10 | −9 | −16 | 15 | −14 | 13 |
| 5 | 6 | 7 | 8 | −1 | −2 | −3 | −4 | −13 | −14 | −15 | −16 | 9 | 10 | 11 | 12 |
| 6 | −5 | −8 | 7 | −2 | 1 | 4 | −3 | −14 | 13 | 16 | −15 | 10 | −9 | −12 | 11 |
| 7 | 8 | −5 | −6 | 3 | 4 | −1 | −2 | −15 | −16 | 13 | 14 | −11 | −12 | 9 | 10 |
| 8 | −7 | 6 | −5 | 4 | −3 | 2 | −1 | −16 | 15 | −14 | 13 | −12 | 11 | −10 | 9 |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | −1 | −2 | −3 | −4 | −5 | −6 | −7 | −8 |
| 10 | −9 | −12 | 11 | 14 | −13 | −16 | 15 | −2 | 1 | 4 | −3 | −6 | 5 | 8 | −7 |
| 11 | 12 | −9 | −10 | −15 | −16 | 13 | 14 | −3 | −4 | 1 | 2 | 7 | 8 | −5 | −6 |
| 12 | −11 | 10 | −9 | −16 | 15 | −14 | 13 | −4 | 3 | −2 | 1 | 8 | −7 | 6 | −5 |
| 13 | 14 | 15 | 16 | −9 | −10 | −11 | −12 | 5 | 6 | 7 | 8 | −1 | −2 | −3 | −4 |
| 14 | −13 | −16 | 15 | −10 | 9 | 12 | −11 | 6 | −5 | −8 | 7 | −2 | 1 | 4 | −3 |
| 15 | 16 | −13 | −14 | 11 | 12 | −9 | −10 | 7 | 8 | −5 | −6 | 3 | 4 | −1 | −2 |
| 16 | −15 | 14 | −13 | 12 | −11 | 10 | −9 | 8 | −7 | 6 | −5 | 4 | −3 | 2 | −1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| −1 | −2 | −3 | −4 | −5 | −6 | −7 | −8 | −9 | −10 | −11 | −12 | −13 | −14 | −15 | −16 |
| −2 | 1 | 4 | −3 | −6 | 5 | 8 | −7 | −10 | 9 | 12 | −11 | −14 | 13 | 16 | −15 |
| −3 | −4 | 1 | 2 | 7 | 8 | −5 | −6 | −11 | −12 | 9 | 10 | 15 | 16 | −13 | −14 |
| −4 | 3 | −2 | 1 | 8 | −7 | 6 | −5 | −12 | 11 | −10 | 9 | 16 | −15 | 14 | −13 |
| −5 | −6 | −7 | −8 | 1 | 2 | 3 | 4 | 13 | 14 | 15 | 16 | −9 | −10 | −11 | −12 |
| −6 | 5 | 8 | −7 | 2 | −1 | −4 | 3 | 14 | −13 | −16 | 15 | −10 | 9 | 12 | −11 |
| −7 | −8 | 5 | 6 | −3 | −4 | 1 | 2 | 15 | 16 | −13 | −14 | 11 | 12 | −9 | −10 |
| −8 | 7 | −6 | 5 | −4 | 3 | −2 | 1 | 16 | −15 | 14 | −13 | 12 | −11 | 10 | −9 |
| −9 | −10 | −11 | −12 | −13 | −14 | −15 | −16 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| −10 | 9 | 12 | −11 | −14 | 13 | 16 | −15 | 2 | −1 | −4 | 3 | 6 | −5 | −8 | 7 |
| −11 | −12 | 9 | 10 | 15 | 16 | −13 | −14 | 3 | 4 | −1 | −2 | −7 | −8 | 5 | 6 |
| −12 | 11 | −10 | 9 | 16 | −15 | 14 | −13 | 4 | −3 | 2 | −1 | −8 | 7 | −6 | 5 |
| −13 | −14 | −15 | −16 | 9 | 10 | 11 | 12 | −5 | −6 | −7 | −8 | 1 | 2 | 3 | 4 |
| −14 | 13 | 16 | −15 | 10 | −9 | −12 | 11 | −6 | 5 | 8 | −7 | 2 | −1 | −4 | 3 |
| −15 | −16 | 13 | 14 | −11 | −12 | 9 | 10 | −7 | −8 | 5 | 6 | −3 | −4 | 1 | 2 |
| −16 | 15 | −14 | 13 | −12 | 11 | −10 | 9 | −8 | 7 | −6 | 5 | −4 | 3 | −2 | 1 |