New $E(s^2)$-Optimal Supersaturated Designs Constructed From Incomplete Block Designs

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We present a method for constructing two-level supersaturated designs (SSDs) from incomplete block designs. A lower bound of $E(s^2)$ that also covers the case of odd run sizes is given. This bound is attained by SSDs constructed from balanced incomplete block designs. We study SSDs that can be constructed from regular graph designs when balanced incomplete block designs do not exist. A computer search is conducted to find SSDs with $5 \le n \le 50$ and $n \le m \le 2n$ that can be constructed from regular graph designs, where m is the number of factors and n is the run size. Many SSDs derived from regular graph designs are optimal. The best $E(s^2)$-optimal SSDs with respect to additional optimality criteria are tabulated. Some notes on the construction of saturated designs also are given.

KEY WORDS: Balanced incomplete-block design; Cyclic incomplete-block design; Plackett–Burman design; Regular graph design; Saturated design.

1. INTRODUCTION

A design for a two-level factor screening experiment is called a *saturated design* (SD) if the number of factors, m, is equal to $n - 1$, where n is the number of runs. Some examples of SDs are Plackett–Burman designs (Plackett and Burman 1946), p-efficient designs (Lin 1993b), and other SDs explored by Dean and Draper (1999) and Crosier (2000). When $m > n - 1$, the design is called a *supersaturated design* (SSD). SSDs were introduced by Booth and Cox (1962) and were not studied further until the important work of Lin (1993a) and Wu (1993). Since then, much work has been done on this subject, recently by Bulutoglu and Cheng (2004), Jones, Lin, and Nachtsheim (2008), Bulutoglu (2007), and Ryan and Bulutoglu (2007). A commonly used criterion for choosing an SSD is the $E(s^2)$ criterion defined by Booth and Cox (1962).

Nguyen (1996) described a passenger-impact crash test on a planned new four–wheel-drive range, where the objective is to find, out of 54 safety features, a subset to be included in new cars' total safety system. He proposed an SSD with $(n, m) = (28, 54)$, which used only 28 car prototypes. We assume that the research and development department of the car manufacturer wants to know whether there is a more economical design.

Consider another study conducted to examine 16 factors affecting the thermal performance of project homes: wall insulation (R1 or R2), roof insulation (R2.5 or R3.5), floor insulation (R0 or R1), floor type (timber or concrete), wall type (brick veneer or cavity), north glass (5% or 20%), east glass (5% or 15%), west glass (5% or 15%), south glass (5% or 15%), east blinds (yes or no), west blinds (yes or no), south blinds (yes or no), north eave overhang (20% or 100%), east eave overhang (20% or 70%), west eave overhang (20% or 70%), and south eave overhang (20% or 100%). Two additional two-level factors are required to set up the blocking factor for homes orientation (north, east, west, or south). The number of homes that can be used for this study is 16. Thus we need an SSD with $(n, m) = (16, 18)$. To keep the prices of these homes comparable, the scientist wants each home to have nine factors at the low level and nine factors at the high level. This constraint makes computer-generated SSDs, such as those described by Nguyen (1996), unsuitable.

This article is motivated by finding combinatorial solutions for these two problems. By exploiting a relationship between SSDs and incomplete block designs (IBDs) presented in Section 3, we show that a solution for the first problem is an $E(s^2)$-optimal SSD (OSSD) with $(n, m) = (27, 54)$ derived from a balanced incomplete-block design (BIBD) with 27 treatments and 54 blocks of size 13, and that a solution for the second problem is an SSD with $(n, m) = (16, 18)$ derived from a regular graph design (RGD) with 16 treatments and 18 blocks of size 8. In Section 2 we review the $E(s^2)$ criterion and present a new lower bound of this criterion that also covers the case where n is odd; previously, lower bounds of $E(s^2)$ were available only for even n's. It follows from this bound that SSDs derived from BIBDs are $E(s^2)$-optimal. In Section 3 we identify a class of RGD-derived SSDs that are $E(s^2)$-optimal when BIBDs do not exist. A computer search to find SSDs with $5 \le n \le 50$ and $n \le m \le 2n$ that can be derived from RGDs is reported in Section 4. Many of these designs have a nice cyclic structure.

2. $E(s^2)$ AS A MEASURE OF GOODNESS OF SUPERSATURATED DESIGNS

Let X be an $n \times m$ design matrix with n runs (rows) and m 2-level factors (columns). We require that $[\frac{1}{2}n]$ of the entries in

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each column be $+1$ and the remainder be -1. Let s_{ij} be the element in the ith row and jth column of $X'X$. A commonly used criterion for comparing SSDs of the same size is the minimization of $E(s^2) = \sum_{i<j} s_{ij}^2 / \binom{m}{2}$.

Nguyen (1996) and Tang and Wu (1997) showed that when n is even,

$$E(s^2) \geq \big(m(n^2+n-1) - n^3\big)/(n(m-1)). \tag{1}$$

This bound is attainable only if n is a multiple 4 and m is a multiple of $n-1$, or $n \equiv 2 (\mathrm{mod}\, 4)$ and m is an even multiple of $n-1$. Improved bounds when (1) is not attainable were obtained by Butler, Mead, Eskridge, and Gilmour (2001) and Bulutoglu and Cheng (2004).

A general bound that also covers the case where n is odd can be derived as follows. For an SSD, the sum of the diagonal elements of XX' is equal to nm. When n is even (resp., odd), because the sum of the elements of XX' is 0 (resp., m), the sum of the off-diagonal elements of XX' is equal to $-mn$ [resp., $-m(n-1)$]. It follows that the sum of squares of the elements of $X'X$, which is also the sum of squares of the elements of XX', reaches the minimum if all of the off-diagonal elements of XX' are equal, that is, XX' is of the form $(m-x)I_n + xJ_{n\times n}$, where $x = -m/(n-1)$ for even n and $-m/n$ for odd n, I_n is the identity matrix of order n, and $J_{n\times n}$ is the $n \times n$ matrix of all 1's. This results in the lower bound

$$E(s^2) \geq n\big(m^2 + (n-1)x^2 - mn\big)/(m(m-1)) \tag{2}$$

for all SSDs. When n is even, (2) is the same as the Nguyen–Tang–Wu bound in (1). For odd n, it leads to the following new bound:

$$E(s^2) \geq \big(m(n^2+n-1) - n^3\big)/(n(m-1)). \tag{3}$$

The lower bound in (3) is attainable only if $n \equiv 3 (\mathrm{mod}\, 4)$ and m is multiple of n, or $n \equiv 1 (\mathrm{mod}\, 4)$ and m is an even multiple of n. Otherwise, suppose that (a) n is odd, m is not a multiple of n, and t is the unique integer such that $-2n < m - tn < 2n$ and $m + t \equiv 2 (\mathrm{mod}\, 4)$, or (b) $n \equiv 1 (\mathrm{mod}\, 4)$ and $m = tn$, where t is odd.

Then the lower bound in (3) can be improved as follows:

$$E(s^2) \geq \big(n(m+t)^2 + 2(n-1)^2 - (tn)^2 - 2tm - mn^2\big) / (m(m-1)). \tag{4}$$

To save space, the proof is omitted.

3. SUPERSATURATED DESIGNS DERIVED FROM INCOMPLETE BLOCK DESIGNS

$E(s^2)$-optimal designs attaining lower bound (2) can be constructed from BIBDs. An equireplicate IBD of size (v, b, k) has v treatments set out in b blocks of size $k\,(< v)$ such that each treatment is replicated r times, where $rv = bk$. We assume that no treatment occurs more than once in a block. Such an equireplicate IBD is a BIBD if every pair of treatments appears together in the same number of blocks. Two important matrices associated with an IBD are the *incidence* matrix $N_{v\times b} = \{n_{ij}\}$ $(i = 1, \ldots, v, j = 1, \ldots, b)$, where n_{ij} equals 1 if treatment i occurs in block j and 0 otherwise, and the *concurrence* matrix $NN' = \{\lambda_{ij}\}$ where, for equireplicate designs, $\lambda_{ii} = r$ $(i = 1, \ldots, v)$ and λ_{ij} $(i \neq j)$ is the number of blocks in which both treatments i and j appear. When the blocks of an IBD are generated by one or more initial blocks, it is called a *cyclic* IBD. More information on IBDs and cyclic IBDs is available in Nguyen (1994) and John and Williams (1995).

From each equireplicate IBD of size (v, b, k) without repeated blocks, where $v \leq b$ and $k = [v/2]$, we can construct a v-run b-factor SSD by letting $X = 2N - J_{v\times b}$, that is, replacing the 0's in N with -1's. For example, the following is the design matrix X of an OSSD with $(n, m) = (14, 14)$ constructed from an IBD of size $(v, b, k) = (14, 14, 7)$:

$$\begin{array}{cccccccccccccc}
+ & - & + & - & + & + & - & + & + & - & + & - & - & - \\
+ & - & - & - & - & - & - & + & + & - & + & + & + & + \\
- & + & + & + & + & - & - & - & + & - & - & + & + & - \\
+ & + & - & - & + & + & + & - & - & - & + & - & + & - \\
- & + & - & - & - & + & + & + & + & + & - & + & - & - \\
- & - & + & - & - & + & - & - & - & + & + & + & + & + \\
- & - & + & - & + & - & + & + & - & + & - & - & + & + \\
+ & - & + & + & - & + & + & - & - & - & - & + & - & + \\
- & - & - & + & + & - & + & + & - & + & + & + & - & - \\
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- & + & + & + & - & - & + & - & + & - & + & - & - & + \\
- & + & - & + & + & + & - & + & - & - & - & - & + & + \\
+ & + & + & + & - & - & - & + & - & + & + & - & - & -
\end{array}$$

Note that the first column of this SSD has positive entries in rows 1, 2, 4, 8, 10, 11, and 14, corresponding to the seven treatments in the first block (1 2 4 8 10 11 14) of the IBD. This IBD is not a BIBD and is not cyclic.

This method of constructing an SSD from an IBD is different from that studied by Nguyen (1996) and Cheng (1997).

We have the following relationship between XX' and NN':

$$XX' = 4NN' + (b - 4r)J_{v\times v}. \tag{5}$$

By (5), minimizing $E(s^2)$ is equivalent to minimizing $\sum \lambda_{ij}^2$ for the IBD. Because $\sum \lambda_{ij}$ is constant $(= vkr)$, $\sum \lambda_{ij}^2$ is minimized if the λ_{ij} $(i \neq j)$ differ by at most 1. John and Mitchell (1977) termed equireplicate IBDs with this property *regular graph designs* (RGDs). The foregoing observation shows that even though an RGD-derived SSD may not be $E(s^2)$-optimal over all SSDs, it is the best of those constructed from equireplicate IBDs and often is the best over all SSDs (see later). When v is even, an SSD constructed from an equireplicate IBD ensures that in each run, half of the factors appear at the high level, and the other half appear at the low level. Thus the constraint given in the second motivating example in Section 1 is satisfied.

The λ_{ij}'s of RGDs are either $\lambda = [r(k-1)/(v-1)]$ or $\lambda + 1$. The second value appears $n_2 = r(k-1) - \lambda(v-1)$ times, and the first value appears $n_1 = v - 1 - n_2$ times in each row of NN'. When $n_2 = 0$, an RGD is also a BIBD. In this case, because NN' is equal to $(r - \lambda)I_n + \lambda J_{n\times n}$, by (5), XX' is of the form $(m-x)I_n + xJ_{n\times n}$. It follows that X attains the lower bound (2) and is $E(s^2)$-optimal. Cheng (1997) showed that for odd v, the design obtained by adding a row of 1's to such a BIBD-derived X is also $E(s^2)$-optimal.

In general, RGD-derived SSDs are not necessarily $E(s^2)$-optimal. The following statements summarize situations in which they are optimal:

A. Suppose that n is even and m is not a multiple of $n-1$. Let t be the unique integer such that $-2(n-1) < m-t(n-1) < 2(n-1)$ and $m+t \equiv 2 (\mathrm{mod}\, 4)$. Then the corresponding RGD is not a BIBD, and its derived SSD attains the Bulutoglu–Cheng bound if

$$n \equiv 0 (\mathrm{mod}\, 4), \qquad m \equiv 2 (\mathrm{mod}\, 4), \qquad \text{and}$$
$$|m - t(n-1)| \geq 3n/2 - 2;$$
$$n \equiv 2 (\mathrm{mod}\, 4) \qquad \text{and} \qquad n = m;$$

or

$$n \equiv 2 (\mathrm{mod}\, 4), \qquad m \equiv 2 (\mathrm{mod}\, 4), \qquad \text{and}$$
$$|m - t(n-1)| \geq 3n/2 - 3.$$

B. For odd n, it can be seen that an RGD-derived SSD exists only if m a multiple of n; so in this case RGD-derived SSDs are less abundant than the case of even n. When $n \equiv 3 (\mathrm{mod}\, 4)$ and m is a multiple of n, or $n \equiv 1 (\mathrm{mod}\, 4)$ and m is an even multiple of n, the corresponding RGD must be a BIBD; therefore, the derived SSD is $E(s^2)$-optimal. In the remaining case where $n \equiv 1 (\mathrm{mod}\, 4)$ and m is an odd multiple of n, the corresponding RGD is not a BIBD, and its derived SSD does not even attain the improved lower bound (4). Such SSDs are not likely to be $E(s^2)$-optimal.

Remark 1. We denote the off-diagonal element in the ith row and jth column of XX' by a_{ij}. The a_{ij}'s of an RGD-derived SSD are either $a_1 = 4\lambda + b - 4r$ or $a_2 = 4(\lambda + 1) + b - 4r$. BIBD-derived SSDs have $a_{ij} = 4\lambda + b - 4r$. If a BIBD is symmetric (i.e., $v = b$), then the off-diagonal elements of $X'X$ also are equal to this value.

Remark 2. For odd n, a BIBD with $v = n, b = m = qn, k = \frac{1}{2}(n-1), r = qk = \frac{q}{2}(n-1)$, and $\lambda = \frac{q}{4}(n-3)$, where q is even if $n \equiv 1 (\mathrm{mod}\, 4)$, corresponds to an SSD with

$$XX' = q(n+1)I_n - qJ_{n\times n}. \tag{6}$$

This XX' matrix is that of an OSSD of size (n, qn). Cyclic solutions are readily available for many of these BIBDs (and the derived OSSDs). Note that because the design matrix of the SSD is obtained by replacing the 0's in the incidence matrix of the block design with -1's, a cyclic construction, if available, applies to both the IBD and SSD.

Remark 3. If we add a row of 1's to the X matrix of an OSSD derived from a BIBD with the parameter combinations in Remark 2 (n is now even), then it also can be shown that the resulting SSD has an XX' matrix having form (6) and is thus an OSSD. OSSDs constructed in this way have been studied by Nguyen (1996) for $q = 2$ and Cheng (1997) and Liu and Zhang (2000) for $q \geq 2$. Note that OSSDs of the same sizes also can be constructed directly from BIBDs with $v = n, b = m = q(n-1), k = \frac{1}{2}n, r = \frac{q}{2}(n-1)$, and $\lambda = \frac{q}{4}(n-2)$ without adding a row of 1's. But because cyclic solutions are not available for BIBDs with these parameter combinations, a computer search of such OSSDs is feasible only for $n \leq 12$.

Remark 4. It is not difficult to show that some results of Nguyen (1996) and Cheng (1997) that apply to the OSSDs obtained from their method of construction (such as that described in Remark 3) also apply to the BIBD-derived SSDs discussed in this article. These results include that deleting a column of a BIBD-derived SSD results in a design with the same $E(s^2)$ and $r_{\max} = \max(|r_{ij}|)$, where r_{ij} is the correlation between columns i and j of X, and that designs obtained by deleting a column of (or adding a column to) a BIBD-derived SSD are $E(s^2)$-optimal.

Remark 5. If we add all of the columns of a BIBD-derived SSD or a BIBD-derived SD, such as a Plackett–Burman design (Plackett and Burman 1946), to an OSSD satisfying the Bulutoglu–Cheng bound, then the resulting design is also an OSSD, provided that it contains no completely aliased columns. This is because if X is the design matrix of an OSSD satisfying the Bulutoglu–Cheng bound and X^* is the design matrix of the new SSD, then $a^*_{ij} = a_{ij} - t$ for some positive integer t, for all off-diagonal entries a_{ij} and a^*_{ij} of XX' and $X^*(X^*)'$. Carefully examining the argument of Bulutoglu and Cheng (2004), especially their (4.2) and (4.3), demonstrates that if X satisfies the Bulutoglu–Cheng bound, then so does X^*.

4. SUPERSATURATED DESIGNS DERIVED FROM REGULAR GRAPH DESIGNS

To construct RGD-derived SSDs with $5 \leq n \leq 50$ and $n \leq m \leq 2n$, we conducted a computer search of RGDs with $5 \leq v \leq 50$, $v \leq b \leq 2v$, and $k = [v/2]$. There are 391 possible parameter combinations (v, b, k) for such SSDs: (a) 11 with $a_{ij} = -1$, (b) 46 with $a_{ij} = -2$, (c) 144 with $a_{ij} = \pm 2$, (d) 178 with $a_{ij} = -4$ or 0, and (e) 12 with $a_{ij} = -3$ or 1. The SSDs in (a) and (b), if they exist, are OSSDs.

SSDs With $a_{ij} = -1$ (a). These SSDs, corresponding to symmetric BIBDs with the parameter combinations in Remark 2 ($q = 1$), are available when $n = m \equiv 3 (\mathrm{mod}\, 4)$. These OSSDs can be obtained by the generating vectors or matrix (for the case $n = 27$) of Plackett and Burman (1946). Note that because these generating vectors contain $[\frac{1}{2}n] + 1$ 1's, a sign change is needed if we want to have $[\frac{1}{2}n]$ 1's in each column of the resulting SSD.

SSDs With $a_{ij} = -2$ (b). We were able to find 32 of these (Table 1). There are 16 OSSDs of size $(n, 2n)$ where n is odd, corresponding to BIBDs with the parameter combinations in Remark 2 ($q = 2$). The remaining OSSDs are of size $(n+1, 2n)$ and also correspond to BIBDs, but, as pointed out in Remark 3 of Section 3, cyclic solutions are not available for these BIBDs. Instead, we can obtain OSSDs by adding a row of 1's to cyclic OSSDs of size $(n, 2n)$ (cf. Remark 3). These OSSDs may differ in the magnitude of $r_{\max}$ as defined in Remark 4, or may have the same $r_{\max}$ but differ in the percentage (%) of r_{ij} having the same absolute value as $r_{\max}$. To construct the generating vectors in Table 1, we generate a large number of cyclic BIBDs and report generating vectors that result in the best OSSDs in terms of minimum $r_{\max}$ and the % of $|r_{ij}| = r_{\max}$. As pointed out in Remark 3, a computer search of OSSDs directly from noncyclic BIBDs is feasible only for $n \leq 12$, in which case we were not able to improve the results in Table 1.

We were able to improve three OSSDs in table 1 of Nguyen (1996) with respect to the additional criteria mentioned in the

Table 1. Generating vectors of OSSDs of size $(n, 2n)$ and $(n+1, 2n)$, obtained by adding a row of 1's to OSSDs of size $(n, 2n)$

n	$E(s^2)^1$	$r_{\max}{}^1$	$E(s^2)^2$	$r_{\max}{}^2$	Generating vectors
5	3.67	.600	4	.333	+ + − − −
		(33.33)§		(100.00)§	+ − + − −
7	4.69	.714	4.92	.500	− + + − − − +
		(7.69)		(30.77)	− + + − + − −
9	5.71	.556	5.88	.600	+ − + + − − − − +
		(5.88)		(5.88)	+ + − + − + − − −
11	6.71	.455	6.86	.333	+ + − − + − − − − + +
		(14.29)		(42.86)	+ − + − − + + − − + −
13	7.72	.385	7.87	.429	+ − − + − − − − + + + − +
		(12.00)		(12.00)	+ − + + + − − − + − + − −
15	8.72	.333	8.83	.250	+ + − − − + − − + − − − + + +
		(20.69)		(55.17)	+ + − − + − + − − − + + − + −
17	9.73	.412	9.82	.333	+ − + + + − − − + + − − + − + − −
		(6.06)		(18.18)	+ + + + − − − + − − + − − − + − +
19	10.73	.263	10.81	.200	+ − + + − − + + − − − + + + − + − − −
		(27.03)		(67.57)	+ + − + − + − − + − − − − − + + + + −
21	11.73	.333	11.81	.273	+ + − − − − − + + − − + − − + − + − + + +
		(7.32)		(24.39)	+ − + − − + − − + − − − + + + − − + + + −
23	12.73	.304	12.80	.333	+ − − − + − − + − + − − + + − + + + − + − − +
		(2.22)		(2.22)	+ + − + − + − − + + + + + − − − − + − − − − +
25	13.74	.280	13.80	.231	+ − + + − + + + + + − − − − − + − − − + + − + − −
		(10.20)		(30.61)	+ − − + + + − + − − − + − − + + − − − + + + − + −
27	14.74	.259	14.79	.286	+ + + + − + − − − − + + − + − − − + − − − + − + + − +
		(7.55)		(7.55)	+ − − + + + − + − + − − + − − − + + − − − + + + + − −
29	15.74	.241	15.78	.200	+ + + − + − − + + + − + + − − − − − − − + − − + + + − + −
		(14.04)		(36.84)	+ + − − + − − + − + + − − − + − + + + + − − + − + − − − +
31	16.74	.290	16.79	.250	+ + + + + − + − − + + − − + + − − − − + − − − + + − + − − + −
		(3.28)		(11.48)	+ − + + + − − − + − + + − − + + − + − + − − − − − − + + + + −
33	17.74	.273	17.78	.294	+ − + − + − − − + + − + + − + − − − − + − + + + + − − + + + − − −
		(1.54)		(1.54)	+ − + − − + + + + + + + − − − + − − + − − − + − − − + + − + − − +
35	18.74	.257	18.78	.222	+ + − − + + + − + + + − − − + − − − + + − + − − − + − − + − + − + + −
		(5.80)		(15.94)	+ + − + − − − + + + + − − + − + + + − + − − − − − + − − − − + + + + −

[1] $E(s^2)$ and $r_{\max}$ for SSDs of size $(n, 2n)$.

[2] $E(s^2)$ and $r_{\max}$ for SSDs of size $(n+1, 2n)$.

§% of $|r_{ij}| = r_{\max}$.

previous paragraph: (20, 38), (24, 46), and (28, 54). The first two also have been reported by Liu and Zhang (2000). All OSSDs with odd n and OSSDs with even $n > 30$ in this table have not been reported elsewhere. It is interesting to note that OSSDs with $n = 9, 13, 23, 27$, and 33 have smaller $r_{\max}$ than the corresponding OSSDs in $n+1$ runs. Our new OSSD of size (27, 54) with $r_{\max} = .259$ can be used as a good alternative to the OSSD of size (28, 54) with $r_{\max} = .286$, used in the passenger-impact crash test experiment mentioned in Section 1.

SSDs With $a_{ij} = \pm 2$ (c). These SSDs have even n and $m \equiv 2 \pmod 4$ and are constructed from RGDs with $v = n$, $b = m$, and $k = \frac{1}{2}v$. According to the result outlined in statement A of Section 3, they are OSSDs if $n = m$, or $n \equiv 0 \pmod 4$ and $m \geq 3n/2 - 2$, or $n \equiv 2 \pmod 4$ and $m \geq 3n/2 - 3$. We found all of the 12 OSSDs with $n = m$. With the exception of the OSSD with $(n, m) = (14, 14)$ displayed in Section 3, they all have cyclic solutions, the generating vectors of which are given in Table 2. Of the remaining RGD-derived SSDs, we were able to find only eight, with $(n, m) = (10, 14), (12, 14), (12, 18), (14, 18), (14, 22), (16, 18), (16, 22)$, and $(16, 26)$. Except for the two SSDs with $(n, m) =$ (12, 14), (16, 18), the other six are OSSDs. The $E(s^2)$, $r_{\max}$, and % of $r_{ij} = r_{\max}$ of these eight SSDs (available at *http://designcomputing.net/ssd/*) are given in Table 3. Note that except for $(n, m) = (14, 22)$, $E(s^2)$-optimal designs for the parameter combinations in Table 3 have been previously reported by Butler et al. (2001) or Bulutoglu and Cheng (2004), but their designs cannot be constructed from RGDs.

SSDs With $a_{ij} = -4$ *or* 0 (d). These SSDs have even n and $m \equiv 0 \pmod 4$ and are constructed from RGDs with $v = n$, $b = m$, and $k = \frac{1}{2}v$. $E(s^2)$ of these SSDs are far from the Bulutoglu–Cheng bound; therefore, we do not make any attempt to construct them.

SSDs With $a_{ij} = -3$ *or* 1 (e). These SSDs are of size $n = m \equiv 1 \pmod 4$ and are constructed from symmetric RGDs with $v \equiv 1 \pmod 4$ and $k = \frac{1}{2}(v-1)$. Although this class of SSDs is not optimal [compared with the bound in (4)], by adding a row of 1's to the design matrices, they can be used to construct SDs with XX' and $X'X$ having off-diagonal elements equal to ± 2. The cyclic solutions together with their D- and A-efficiencies (calculated with a column of 1's in the model matrix) are given in Table 4. These efficiencies improve on those reported in table

Table 2. Generating vectors of OSSDs of size (n, n) where $n \equiv 2 (\bmod 4)$

n	r_{max}	Generating vectors
6	$\frac{1}{3}$	+−+−−+
10	$\frac{1}{5}$	+−−−−++−++
18	$\frac{1}{9}$	++−+−+−−+−−++−−−++
22	$\frac{1}{11}$	+−+−+++−++−−−−+++−−+−−
26	$\frac{1}{13}$	+−−+−+−−−−+++++−++−+−+−−−+
30	$\frac{1}{15}$	+++++−+−−−+++−++−−+−+−−−−+−+−−
34	$\frac{1}{17}$	+−+−−++++−−−−−−+−+−−+−+−+++−+++−−+
38	$\frac{1}{19}$	+−+++++−−+++++−−+−−−−+−+−−++−+−+−−−+−−
42	$\frac{1}{21}$	+−+−+−+−−++−+−−++−−−−+−−−−++++++−−++−+++−−
46	$\frac{1}{23}$	+−−−+++++−+−+−+−−+−−−−++−−++−+−++−−−−−+++−++−+
50	$\frac{1}{25}$	+++−−−−+−−++++−+++++−+−−+−+−+−++−−−−+−−++−−++−+−−−

NOTE: These OSSDs have $E(s^2) = 4$.

Table 3. $E(s^2)$ and r_{max} of SSDs constructed from noncyclic RGDs

n	m	$E(s^2)$	Bound[1]	r_{max} (%)[2]
10	14	5.0549	5.0549	.600 (3.30)
12	14	4.7473	4.2198	.333 (29.67)
12	18	5.9608	5.9608	.333 (37.25)
14	18	5.6732	5.6732	.429 (5.23)
14	22	6.9091	6.9091	.429 (9.09)
16	18	5.0196	4.1830	.250 (31.37)
16	22	6.6494	6.6494	.500 (.87)
16	26	7.8769	7.8769	.500 (1.85)

[1] The Bulutoglu–Cheng bound.
[2] % of $|r_{ij}| = r_{max}$.

10 of Lin (1993b) and table 2 of Crosier (2000). For $n \leq 30$, these efficiencies match those in tables 3 and 4 of Dean and Draper (1999).

In cases (d) and (e), optimal designs generally must come from outside the class of RGD-derived designs. The bound in (4) is sharp in certain cases; for example, we have found OSSDs achieving the bound in (4) for $(n, m) = (5, 5), (7, 8), (7, 9), (9, 9)$, and $(13, 13)$.

5. CONCLUDING REMARKS

IBDs are related to several combinatorial structures, including Hadamard matrices (Plackett and Burman 1946), Box–Behnken designs (Box and Behnken 1960; Nguyen and Borkowski 2008), SDs (Dean and Draper 1999; Crosier 2000) and SSDs (Nguyen 1996; Cheng 1997; Liu and Zhang 2000). This article studies additional relationship between IBDs and SSDs. The idea is to construct more complex combinatorial structures from simpler ones.

Many BIBDs and RGDs have cyclic solutions. To construct OSSDs from cyclic BIBDs, we need to be aware that the obtained OSSDs might differ in r_{max} and the percent of $|r_{ij}|$ having the same value as r_{max}. Our approach is to generate a large number of candidate OSSDs from cyclic BIBDs and retain the one with minimum r_{max} (and percent of $|r_{ij}| = r_{max}$). This approach should be used even for OSSDs of moderate run sizes.

Using the result in Remark 5 of Section 3, we can use the BIBD-derived SSDs tabulated in this article as a base for constructing larger SSDs using the algorithm of Nguyen (1996).

Table 4. Generating vectors of saturated designs for $n \equiv 2 (\bmod 4)$

n	E_D	E_A	Generating vectors
6	.7631	.5455	++−−−
10	.8658	.7510	++−+−−−−+
14	.9046	.8251	+−++−+−−−−−++
18	.9258	.8647	+−−−+−+−−+−−−++++
22	.9360	.8834	++++−−−−−+−−++−−++−+−
26	.9365	.8774	+−−−−−+++++−−++−−++−+−−+−
30	.9477	.9008	+++−++−−−+++−+−−−+−+−−−+−−+−+
34	.9511	.9062	+−++++−−−+++−−−+−−−+−−++−++−+−−+−
38	.9558	.9158	+−−++−+++−−−−−+−+−−−−+++−+−++−++++−−−
42	.9578	.9187	+−−+++−−−−−−−++−+−−++−+−+−+−−+−++++−−++−+
46	.9664	.9382	+−−−+−−−−−−++++++−−+++−−+++−−+−++−++−−+−−+−+−
50	.9657	.9338	+−−−−+−−+−−+−++−−++−−+−+−+++++−−−+−++++−+−+++−−−−

NOTE: The final design includes a row of 1's.

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